

Early Universe: inflation and cosmological perturbations

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Abstract

After a brief summary of general relativity and cosmology, we present the basic concepts underlying inflation, the currently best motivated models for the early Universe. We describe the simplest inflation models, based on a single scalar field, and discuss how primordial cosmological perturbations are generated. We then review some recent developments concerning multi-field inflation models, in particular multi-field Dirac-Born-Infeld inflation.

1 Introduction

Inflation, i.e. a phase of accelerated expansion, has now become a standard paradigm to describe the physics of the very early universe. Although many models of inflation have now been ruled out by observations, many remain compatible with the present data and the nature of the field(s) responsible for inflation is still an open question. As more and more precise cosmological data will continue to accumulate in the coming years, one can envisage the fascinating possibility to learn from the observations some crucial clues about the fundamental physics at work in the very early Universe.

In the first part of this contribution, we present a few basic results from general relativity and from standard cosmology. The second part is devoted to the simplest models of inflation and to the computation of the cosmological perturbations that these models generate, which is crucial for confrontation with cosmological observations. These first two parts are mainly based on the pedagogical introduction [1] where the reader will find more details and references.

In the third part, we go beyond the simplest models by extending our analysis to models of inflation involving several scalar fields. We show how the standard results are modified in this context and discuss various models which have attracted a lot of attention during the last years. We present recent results

for very general multi-field inflationary models, allowing for non-standard kinetic terms. This generalization is motivated by efforts to connect string theory and inflation and we focus our attention on multi-field DBI (Dirac-Born-Infeld) inflation.

2 A few elements on general relativity and cosmology

2.1 General relativity

The standard model of modern cosmology is based on Einstein's theory of general relativity. Without entering into details, which can be found in standard textbooks on general relativity, let us recall a few useful notions. In the framework of general relativity, the spacetime geometry is defined by a *metric*, a symmetric tensor with two indices, whose components in a coordinate system $\{x^\mu\}$ ($\mu = 0, 1, 2, 3$) will be denoted $g_{\mu\nu}$. The square of the “distance” between two neighbouring points of spacetime is given by the expression

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (1)$$

We will use the signature $(-, +, +, +)$.

It is convenient to define a *covariant derivative* associated to this metric, denoted ∇_μ , whose action on a tensor with, for example, one covariant index and one contravariant index will be given by

$$\nabla_\lambda T^\mu{}_\nu = \partial_\lambda T^\mu{}_\nu + \Gamma^\mu_{\lambda\sigma} T^\sigma{}_\nu - \Gamma^\sigma_{\lambda\nu} T^\mu{}_\sigma \quad (2)$$

(a similar term must be added for each additional covariant or contravariant index), where the Γ are the *Christoffel symbols* (they are not tensors), defined by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}). \quad (3)$$

We have used the notation $g^{\mu\nu}$ which corresponds, for the metric (and only for the metric), to the inverse of $g_{\mu\nu}$ in a matricial sense, i.e. $g_{\mu\sigma} g^{\sigma\nu} = \delta_\mu^\nu$.

The “curvature” of spacetime is characterized by the *Riemann* tensor, whose components can be expressed in terms of the Christoffel symbols according to the expression

$$R_{\lambda\mu\nu}{}^\rho = \partial_\mu \Gamma^\rho_{\lambda\nu} - \partial_\lambda \Gamma^\rho_{\mu\nu} + \Gamma^\sigma_{\lambda\nu} \Gamma^\rho_{\sigma\mu} - \Gamma^\sigma_{\mu\nu} \Gamma^\rho_{\sigma\lambda}. \quad (4)$$

Einstein's equations relate the spacetime geometry to its matter content. The geometry appears in Einstein's equations via the *Ricci tensor*, defined by

$$R_{\mu\nu} = R_{\mu\sigma\nu}{}^\sigma, \quad (5)$$

and the *scalar curvature*, which is the trace of the Ricci tensor, i.e.

$$R = g^{\mu\nu} R_{\mu\nu}. \quad (6)$$

The matter enters Einstein's equations via the *energy-momentum tensor*, denoted $T_{\mu\nu}$, whose time/time component corresponds to the energy density, the time/space components to the momentum density and the space/space component to the stress tensor. Einstein's equations then read

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (7)$$

where the tensor $G_{\mu\nu}$ is called the *Einstein tensor*. Since, by construction, the Einstein tensor satisfies the identity $\nabla_\mu G^\mu_\nu = 0$, any energy-momentum on the right-hand side of Einstein's equation must necessarily satisfy the relation

$$\nabla_\mu T^\mu_\nu = 0, \quad (8)$$

which can be interpreted as a generalization, in the context of a curved space-time, of the familiar conservation laws for energy and momentum.

Einstein's equations can also be obtained from a variational principle. The corresponding action reads

$$\mathcal{S} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda) + \int d^4x \sqrt{-g} \mathcal{L}_{mat}. \quad (9)$$

One can check that the variation of this action with respect to the metric $g_{\mu\nu}$, upon using the definition

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_{mat})}{\delta g_{\mu\nu}}, \quad (10)$$

indeed gives Einstein's equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (11)$$

This is a slight generalization of Einstein's equations (7) that includes a *cosmological constant* Λ . It is worth noticing that the cosmological constant can also be interpreted as a particular energy-momentum tensor of the form $T_{\mu\nu} = -(8\pi G)^{-1} \Lambda g_{\mu\nu}$.

2.2 Standard cosmology

Let us now present briefly the tenets of modern cosmology. They are based on Einstein's equations and on a few hypotheses concerning spacetime and its matter content. The most important one, so far confirmed by observations on large scales, is that our universe is approximately homogeneous and isotropic. Geometries that are strictly homogeneous and isotropic are described by the so-called *Robertson-Walker* metrics, which read in an appropriate coordinate system

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (12)$$

with $\kappa = 0, -1, 1$ depending on the curvature of spatial hypersurfaces: respectively flat, elliptic or hyperbolic.

The matter content compatible with the spacetime symmetries of homogeneity and isotropy is necessarily described by an energy-momentum tensor of the form (in the same coordinate system as for the metric (12)),

$$T^\mu_\nu = \text{Diag}(-\rho(t), p(t), p(t), p(t)), \quad (13)$$

where ρ corresponds to an energy density and p to a pressure.

Substituting the Robertson-Walker metric (12) in Einstein's equations (7), one gets the *Friedmann-Lemaître equations*:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2}, \quad (14)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \quad (15)$$

An immediate consequence of these two equations is the *continuity equation*

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (16)$$

where $H \equiv \dot{a}/a$ is the *Hubble parameter*. The continuity equation can be also obtained directly from the energy-momentum conservation $\nabla_\mu T^\mu_\nu = 0$.

In order to determine the cosmological time evolution, it is easier to combine (14) with (16). Let us assume an equation of state for the cosmological matter of the form $p = w\rho$ with w constant, which includes the two main types of matter that play an important rôle in cosmology, namely a gas of relativistic particles, with $w = 1/3$, and non-relativistic matter, with $w \simeq 0$. In these cases, the conservation equation (16) can be integrated to give

$$\rho \propto a^{-3(1+w)}. \quad (17)$$

Substituting in (14), one finds, for $\kappa = 0$,

$$3\frac{\dot{a}^2}{a^2} = 8\pi G\rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}, \quad (18)$$

where, by convention, the subscript '0' stands for *present* quantities. This implies for the evolution of the scale factor

$$a(t) \propto t^{\frac{2}{3(1+w)}} \quad (19)$$

which thus gives $a(t) \propto t^{2/3}$ in a universe dominated by non-relativistic matter and $a(t) \propto t^{1/2}$ in a universe dominated by radiation.

One can also mention the case of a *cosmological constant*, which corresponds to an equation of state $w = -1$ and thus implies an exponential evolution for the scale factor

$$a(t) \propto \exp(Ht). \quad (20)$$

More generally, when several types of matter coexist with respectively $p_{(i)} = w_{(i)}\rho_{(i)}$, it is convenient to introduce the dimensionless parameters

$$\Omega_{(i)} = \frac{8\pi G\rho_0^{(i)}}{3H_0^2}, \quad (21)$$

which express the *present* ratio of the energy density of the various species with the so-called *critical energy density* $\rho_{crit} = 3H_0^2/(8\pi G)$, which corresponds to the total energy density for a flat universe.

One can then rewrite the first Friedmann equation (14) as

$$\left(\frac{H}{H_0}\right)^2 = \sum_i \Omega_{(i)} \left(\frac{a}{a_0}\right)^{-3(1+w_{(i)})} + \Omega_\kappa \left(\frac{a}{a_0}\right)^{-2}, \quad (22)$$

with $\Omega_\kappa = -\kappa/a_0^2 H_0^2$, which implies that the cosmological parameters must satisfy the consistency relation

$$\sum_i \Omega_{(i)} + \Omega_\kappa = 1. \quad (23)$$

As for the second Friedmann equation (15), it implies

$$\frac{\ddot{a}_0}{a_0 H_0^2} = -\frac{1}{2} \sum_i \Omega_{(i)} (1 + w_{(i)}). \quad (24)$$

Cosmological observations yield for the various parameters (see e.g. [2])

- Baryons: $\Omega_b \simeq 0.05$,
- Dark matter: $\Omega_d \simeq 0.23$,
- Dark energy (compatible with a cosmological constant): $\Omega_\Lambda \simeq 0.72$,
- Photons: $\Omega_\gamma \simeq 5 \times 10^{-5}$.

Observations have not detected so far any deviation from flatness. Radiation is very subdominant today but extrapolating backwards in time, radiation was dominant in the past since its energy density scales as $\rho_\gamma \propto a^{-4}$ in contrast with non-relativistic matter ($\rho_m \propto a^{-3}$). Moreover, the matter content today seems to be dominated by some dark energy with a present equation of state very close to that of a cosmological constant ($w_\Lambda = -1$), which means that our universe is currently accelerating.

To go beyond a purely geometrical description of cosmology, it is useful to apply thermodynamics to the matter content of the universe. One can then define a temperature T for the cosmological photons, not only when they are strongly interacting with ordinary matter but also after they have decoupled because, with the expansion, the thermal distribution for the gas of photons is

unchanged except for a global rescaling of the temperature so that T essentially evolves as

$$T(t) \propto \frac{1}{a(t)}. \quad (25)$$

This means that, going backwards in time, the universe was much hotter than today. This is the essence of the Big Bang scenario.

As the universe evolves, the reaction rates between the various species are modified. A detailed analysis of these changes allows to reconstruct the past thermal history of the universe. Two events in particular play an essential rôle because of their observational consequences:

- Primordial nucleosynthesis

Nucleosynthesis occurred at a temperature around 0.1 MeV, when the average kinetic energy became sufficiently low so that nuclear binding was possible. Protons and neutrons could then combine, which lead to the production of light elements, such that Helium, Deuterium, Lithium, etc... Within the observational uncertainties, this scenario is remarkably confirmed by the present measurements.

- Decoupling of baryons and photons (or last scattering)

A more recent event is the so-called “recombination” of nuclei and electrons to form atoms. This occurred at a temperature of the order of the eV. Free electrons thus almost disappeared, which entailed an effective decoupling of the cosmological photons from ordinary matter. What we see today as the Cosmic Microwave Background (CMB) is made of the fossil photons, which interacted for the last time with matter at the time of recombination. The CMB represents a remarkable observational tool for analysing the perturbations of the early universe, as well as for measuring the cosmological parameters introduced above.

2.3 Puzzles of the standard Big Bang scenario

The standard Big Bang scenario has encountered remarkable successes, in particular with the nucleosynthesis scenario and the prediction of the CMB, and it remains today a cornerstone in our understanding of the present and past universe. However, a few intriguing facts remain unexplained in the basic Big Bang model and seem to necessitate a larger framework. Concerning the present matter state of the Universe, we need of course to understand the nature of dark matter and of dark energy. In addition to these two crucial questions, several properties of our Universe are problematic in the perspective of its evolution. We now review these problems.

- Homogeneity problem

A first question is why the approximation of homogeneity and isotropy turns out to be so good. Indeed, inhomogeneities are unstable, because of gravitation, and they tend to grow with time. It can be verified for

instance with the CMB that inhomogeneities were much smaller at the last scattering epoch than today. One thus expects that these homogeneities were still smaller further back in time. How to explain a universe so smooth in its past ?

- Flatness problem

Another puzzle is the (spatial) flatness of our universe. Indeed, Friedmann's equation (14) implies

$$\Omega - 1 \equiv \frac{8\pi G\rho}{3H^2} - 1 = \frac{\kappa}{a^2 H^2}. \quad (26)$$

In standard cosmology, the scale factor behaves like $a \sim t^q$ with $q < 1$ ($q = 1/2$ for radiation and $q = 2/3$ for non-relativistic matter). As a consequence, $(aH)^{-2}$ grows with time and $|\Omega - 1|$ must thus diverge with time. Therefore, in the context of the standard model, the quasi-flatness observed today requires an extreme fine-tuning of Ω near 1 in the early universe.

- Horizon problem

One of the most fundamental problems in standard cosmology is certainly the *horizon problem*. The (particle) *horizon* is the maximal distance that can be covered by a light ray. For a light-like radial trajectory $dr = a(t)dt$ and the horizon is thus given by

$$d_H(t) = a(t) \int_{t_i}^t \frac{dt'}{a(t')} = a(t) \frac{t^{1-q} - t_i^{1-q}}{1-q}, \quad (27)$$

where the last equality is obtained by assuming $a(t) \sim t^q$ and t_i is some initial time.

In standard cosmology ($q < 1$), the integral converges in the limit $t_i = 0$ and the horizon has a finite size, of the order of the so-called Hubble radius H^{-1} :

$$d_H(t) = \frac{q}{1-q} H^{-1}. \quad (28)$$

It is also useful to consider the *comoving Hubble radius*, $(aH)^{-1}$, which represents the fraction of comoving space in causal contact. One finds that it *grows* with time, which means that the *fraction of the universe in causal contact increases with time* in the context of standard cosmology. But the CMB tells us that the Universe was quasi-homogeneous at the time of last scattering on a scale encompassing many regions a priori causally independent. How to explain this ?

A solution to the horizon problem and to the other puzzles is provided by the inflationary scenario, which we will examine in the next section. The basic idea is to invert the behaviour of the comoving Hubble radius, that is to make

it *decrease* sufficiently in the very early universe. The corresponding condition is that

$$\ddot{a} > 0, \quad (29)$$

i.e. that the Universe must undergo a *phase of acceleration*.

3 Single field inflation

The simplest inflationary models are based on a single scalar field ϕ governed by an action of the form

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \right), \quad (30)$$

where $V(\phi)$ is the potential for the scalar field. The corresponding energy-momentum tensor is given by

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} \partial^\sigma \phi \partial_\sigma \phi + V(\phi) \right). \quad (31)$$

3.1 Homogeneous evolution

In a spatially flat FLRW (Friedmann-Lemaitre-Robertson-Walker) spacetime, with metric

$$ds^2 = -dt^2 + a^2(t) d\vec{x}^2, \quad (32)$$

the energy-momentum tensor reduces to the perfect fluid form with energy density and pressure given respectively by

$$\rho = -T_0^0 = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p = \frac{1}{2} \dot{\phi}^2 - V(\phi). \quad (33)$$

The equation of motion for the scalar field is

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0. \quad (34)$$

and the evolution of the scale factor is governed by Friedmann's equations

$$H^2 = \frac{8\pi G}{3} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \quad \dot{H} = -4\pi G \dot{\phi}^2. \quad (35)$$

If the potential satisfies the so-called slow-roll conditions,

$$\epsilon_V \equiv \frac{m_P^2}{2} \left(\frac{V'}{V} \right)^2 \ll 1, \quad \eta_V \equiv m_P^2 \frac{V''}{V} \ll 1, \quad (36)$$

where $m_P \equiv (8\pi G)^{-1/2}$ is the reduced Planck mass, the evolution can enter into a *slow-roll inflationary regime* where the kinetic energy of the scalar field in (35) and the acceleration $\ddot{\phi}$ in the Klein-Gordon equation (34) can be neglected. In

this regime, the equations governing the evolution of the scale factor and the scalar field reduce to

$$H^2 \simeq \frac{8\pi G}{3}V, \quad 3H\dot{\phi} + V' \simeq 0. \quad (37)$$

A useful quantity, which can then be easily derived, is the number of e-folds $N \equiv \ln(a_{\text{end}}/a)$ between some instant during inflation and the end of inflation (or, more precisely, the end of the slow-roll regime):

$$N(\phi) \simeq \int_{\phi}^{\phi_{\text{end}}} \frac{V}{m_P^2 V'} d\phi \quad (38)$$

For any model of inflation, the number of e-folds between the onset of inflation and reheating must be sufficient, typically of the order of 60, in order to solve the horizon problem discussed earlier. In order to get more detailed constraints on the models from observations, it is necessary to go beyond the homogeneous description and consider cosmological perturbations.

3.2 Cosmological perturbations

In the theory of linear cosmological perturbations¹, both the matter (i.e. the scalar field for inflation) and the geometry, i.e. the metric, are perturbed. Restricting ourselves to scalar perturbations, the metric can be written as

$$ds^2 = -(1+2A)dt^2 + 2a(t)\partial_i B dx^i dt + a^2(t) [(1-2\psi)\delta_{ij} + 2\partial_i \partial_j E] dx^i dx^j, \quad (39)$$

where ψ is directly related to the intrinsic curvature of constant time hypersurfaces, according to the relation

$$^{(3)}R = \frac{4}{a^2} \partial^2 \psi. \quad (40)$$

The metric perturbations are modified in a change of coordinates. It is thus useful (although not necessary) to define gauge-invariant quantities, such as the curvature perturbation on uniform energy hypersurfaces, defined by

$$-\zeta \equiv \psi + \frac{H}{\dot{\rho}} \delta\rho = \psi - \frac{\delta\rho}{3(\rho+p)}, \quad (41)$$

or the comoving curvature perturbation,

$$\mathcal{R} \equiv \psi - \frac{H}{\rho+p} \delta q, \quad (42)$$

where δq is the scalar part of the momentum density ($\delta T_i^0 \equiv \partial_i \delta q$). Using the linearized Einstein's equations, it can be shown that these two quantities are related via

$$\zeta = -\mathcal{R} - \frac{2\rho}{3(\rho+p)} \left(\frac{k}{aH} \right)^2 \Psi \quad (43)$$

¹See [1] for a basic presentation and [5] for a detailed review.

where

$$\Psi = \psi + a^2 H(\dot{E} - B/a). \quad (44)$$

The quantity ζ is particularly interesting because it is conserved on large scales when the matter perturbations are adiabatic, i.e. when they satisfy

$$\delta P_{\text{nad}} \equiv \delta p - \frac{\dot{p}}{\dot{\rho}} \delta \rho = 0. \quad (45)$$

This property, which is well-known for linear perturbations, can be seen as the consequence of a more general result. Indeed, the conservation of the energy-momentum tensor for any perfect fluid, characterized by the energy density ρ , the pressure p and the four-velocity u^a , leads to the *exact* relation[3, 4]

$$\dot{\zeta}_a \equiv \mathcal{L}_u \zeta_a = -\frac{\Theta}{3(\rho + p)} \left(\nabla_a p - \frac{\dot{p}}{\dot{\rho}} \nabla_a \rho \right), \quad (46)$$

where we have defined

$$\zeta_a \equiv \nabla_a \alpha - \frac{\dot{\alpha}}{\dot{\rho}} \nabla_a \rho, \quad \Theta = \nabla_a u^a, \quad \alpha = \frac{1}{3} \int d\tau \Theta, \quad (47)$$

and where a dot on scalar quantities denotes here a derivative along u^a (e.g. $\dot{\rho} \equiv u^a \nabla_a \rho$). This identity is valid for any spacetime geometry and does not rely on Einstein's equations. In the cosmological context, α can be interpreted as a non-linear generalization, according to an observer following the fluid, of the number of e-folds of the scale factor. Introducing an explicit coordinate system and linearizing (46) leads to the familiar result of the linear theory.

During inflation, it is easier to work with the perturbation \mathcal{R} , since in this case

$$\mathcal{R} = \psi + \frac{H}{\dot{\phi}} \delta \phi. \quad (48)$$

Because of the constraints arising from Einstein's equations, the scalar metric perturbations and the scalar field perturbation are not independent. In fact, there is only one degree of freedom which can be expressed in terms of the combination

$$v = a \left(\delta \phi + \frac{\dot{\phi}}{H} \psi \right) \equiv a Q, \quad (49)$$

where Q represents the scalar field perturbation in the spatially flat gauge (where $\psi = 0$). The quadratic action governing the dynamics of this degree of freedom can be obtained from the expansion up to second order of the full action. One finds (see e.g. [5])

$$S_v = \frac{1}{2} \int d\tau d^3x \left[v'^2 + \partial_i v \partial^i v + \frac{z''}{z} v^2 \right], \quad (50)$$

where a prime denotes a derivative with respect to the conformal time $\tau = \int dt/a(t)$, and with

$$z = a \frac{\dot{\phi}}{H}. \quad (51)$$

To quantize this system, one considers v as a quantum field and one decomposes it as

$$\hat{v}(\tau, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left\{ \hat{a}_{\vec{k}} v_k(\tau) e^{i\vec{k} \cdot \vec{x}} + \hat{a}_{\vec{k}}^\dagger v_k^*(\tau) e^{-i\vec{k} \cdot \vec{x}} \right\}, \quad (52)$$

where the \hat{a}^\dagger and \hat{a} are creation and annihilation operators, which satisfy the usual commutation rules

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}^\dagger] = \delta(\vec{k} - \vec{k}'), \quad [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = [\hat{a}_{\vec{k}}^\dagger, \hat{a}_{\vec{k}'}^\dagger] = 0. \quad (53)$$

The action implies that the conjugate momenta for v is v' . Therefore, the canonical quantization for \hat{v} and its conjugate momentum leads to the condition

$$v_k v_k'^* - v_k^* v_k' = i. \quad (54)$$

The complex function $v_k(\tau)$ satisfies the equation of motion

$$v'' + \left(k^2 - \frac{z''}{z} \right) v = 0. \quad (55)$$

In the slow-roll limit, $z''/z \simeq 2/\tau^2$, and one can use the solution for a de Sitter spacetime (where H is constant). Note that this is only an approximation as the Hubble parameter is decreasing with time, but a very good one, when the slow-roll parameters are small, during the short time when the scale of interest crosses out the Hubble radius ($k \sim aH$). Requiring that the solution on small scales behaves like the Minkowski vacuum selects the particular solution

$$v_k \simeq \frac{1}{\sqrt{2k}} e^{-ik\tau} \left(1 - \frac{i}{k\tau} \right), \quad (56)$$

where the normalization is imposed by the condition (54). This implies that the power spectrum of the scalar field fluctuations is given by

$$\mathcal{P}_Q = \frac{k^3}{2\pi^2} |v_k|^2 \frac{1}{a^2} \simeq \frac{H^2}{4\pi^2}, \quad (57)$$

where the quantities on the right hand side are evaluated at *Hubble crossing*. This can be translated into the power spectrum of the curvature perturbation \mathcal{R} , by noting that $\mathcal{R} = aQ/z$. One thus gets

$$\mathcal{P}_{\mathcal{R}} = \frac{k^3}{2\pi^2} \frac{|v_k|^2}{z^2} \simeq \left(\frac{H^4}{4\pi^2 \dot{\phi}^2} \right)_{|k=aH} = \frac{1}{2m_P^2 \epsilon_*} \left(\frac{H_*}{2\pi} \right)^2, \quad (58)$$

where ϵ_* is the slow-roll parameter defined in (36), the label $*$ denoting its value at Hubble crossing.

In single-field inflation, since \mathcal{R} is conserved on large scales (as \mathcal{R} and ζ coincide on large scales), the above expression, evaluated at Hubble crossing, determines the amplitude of the curvature perturbation just before the modes reenter the Hubble radius and thus sets the initial conditions for cosmological

perturbations. As we will see in the next section, this is no longer true for inflationary models involving several scalar fields.

We have focused so far our attention on scalar perturbations, which are the most important in cosmology. Tensor perturbations, or primordial gravitational waves, if ever detected in the future, would be a remarkable probe of the early universe. In the inflationary scenario, like scalar perturbations, primordial gravitational waves are generated from vacuum quantum fluctuations. Let us now explain briefly this mechanism.

Starting from the metric with tensor perturbations,

$$ds^2 = a^2(\tau) [-d\tau^2 + (\delta_{ij} + \bar{E}_{ij}) dx^i dx^j], \quad (59)$$

where \bar{E}_{ij} is transverse traceless (i.e. $\partial^i \bar{E}_{ij} = 0$ and $\delta^{ij} \bar{E}_{ij} = 0$), the action expanded at second order in the perturbations yields

$$S_g^{(2)} = \frac{1}{64\pi G} \int d\tau d^3x a^2 \eta^{\mu\nu} \partial_\mu \bar{E}_j^i \partial_\nu \bar{E}_i^j, \quad (60)$$

where $\eta^{\mu\nu}$ denotes the Minkowski metric. Apart from the tensorial nature of \bar{E}_j^i , this action is quite similar to that of a massless scalar field in a FLRW universe, up to a renormalization factor $1/\sqrt{32\pi G}$. The decomposition

$$a\bar{E}_j^i = \sum_{\lambda=+, \times} \int \frac{d^3k}{(2\pi)^{3/2}} v_{k,\lambda}(\tau) \epsilon_j^i(\vec{k}; \lambda) e^{i\vec{k} \cdot \vec{x}} \quad (61)$$

where the $\epsilon_j^i(\vec{k}; \lambda)$ are the polarization tensors, shows that the gravitational waves are essentially equivalent to two massless scalar fields (for each polarization) $\phi_\lambda = m_P \bar{E}_\lambda / 2$.

The total power spectrum is thus immediately deduced from (57):

$$\mathcal{P}_T = 2 \times \frac{4}{m_P^2} \times \left(\frac{H}{2\pi} \right)^2, \quad (62)$$

where the first factor comes from the two polarizations, the second from the renormalization with respect to a canonical scalar field, the last term being the power spectrum for a scalar field derived earlier. In summary, the tensor power spectrum is

$$\mathcal{P}_T = \frac{2}{\pi^2} \left(\frac{H_*}{m_P} \right)^2, \quad (63)$$

where the label $*$ recalls that the Hubble parameter, which can be slowly evolving during inflation, must be evaluated when the relevant scale crossed out the Hubble radius.

4 Multi-field inflation

So far, the simplest models of inflation are compatible with the data but it is instructive to study more refined models for at least two reasons. First,

because models inspired by high energy physics are usually more complicated than the simplest phenomenological inflationary models. Second, because these generalized models will give us an idea of how much the future data will be able to pin down some specific region in the “space” of models.

In this section, we first discuss some potentially observational signatures of more sophisticated models, namely entropic perturbations and non-Gaussianities, which, if ever detected, would provide invaluable additional clues on the early Universe. We then turn to some specific scenarios: the curvaton mechanism, and multi-field inflation with non standard kinetic terms illustrated by multi-field Dirac-Born-Infeld inflation.

4.1 Adiabatic and entropic perturbations

Before considering various types of multi-field scenarios, it is instructive to discuss potentially observational effects that would discriminate between multi-field and single-field inflation. In the case of single field inflation, all perturbations of the cosmological fluid, which consists of photons, neutrinos, baryons and cold dark matter (CDM) particles ultimately originate from the primordial scalar field fluctuations and satisfy the adiabaticity property, $\delta(n_m/n_r) = 0$, or

$$\frac{\delta\rho_m}{\rho_m} = \frac{3\delta\rho_r}{4\rho_r}, \quad (64)$$

where the index m stands for a non-relativistic species (either baryonic matter or CDM) and r for a relativistic species (photons or neutrinos).

By contrast, in a multi-field scenario, one can envisage a richer spectrum of possibilities, such as the existence of non-adiabatic, or entropic perturbations, for example between CDM and photons, defined by

$$S \equiv \frac{\delta\rho_c}{\rho_c} - \frac{3}{4} \frac{\delta\rho_\gamma}{\rho_\gamma}. \quad (65)$$

Interestingly, this entropic perturbation could be correlated with the adiabatic perturbation [6, 7]. The adiabatic and entropic perturbations lead to a different peak structure in the CMB fluctuations and, therefore, CMB measurements can potentially distinguish between these two types of perturbations. On large angular scales, one can show for instance that [6]

$$\frac{\delta T}{T} \simeq \frac{1}{5} (\mathcal{R} - 2\mathcal{S}). \quad (66)$$

The combined impact of adiabatic and entropic perturbations crucially depends on their correlation

$$\beta = \frac{\mathcal{P}_{\mathcal{S},\mathcal{R}}}{\sqrt{\mathcal{P}_{\mathcal{S}}\mathcal{P}_{\mathcal{R}}}}. \quad (67)$$

Parametrizing the relative amplitude between the two types of perturbations by a coefficient α ,

$$\frac{\mathcal{P}_{\mathcal{S}}}{\mathcal{P}_{\mathcal{R}}} \equiv \frac{\alpha}{1 - \alpha}, \quad (68)$$

the present constraints on the entropy contribution are $\alpha_0 < 0.067$ (95% C.L.) in the uncorrelated case ($\beta = 0$) and $\alpha_{-1} < 0.0037$ (95% C.L.) in the totally anti-correlated case ($\beta = -1$) [2].

4.2 Non-Gaussianities

Another interesting feature of some early Universe models is to produce primordial perturbations with a significant non-Gaussianity, which could be detected in future observations (see [8] for a review). Note that, in contrast with entropic perturbations, a significant non-Gaussianity is not specific to multi-field models as single field models with non-standard kinetic terms can produce a (relatively) high level of non-Gaussianity.

The most natural estimate of non-Gaussianity is the bispectrum defined, in Fourier space, by

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle \equiv (2\pi)^3 \delta^{(3)}(\sum_i \mathbf{k}_i) B_\zeta(k_1, k_2, k_3). \quad (69)$$

Equivalently, one often uses the so-called f_{NL} parameter, which can be defined in general by

$$\frac{6}{5} f_{NL} \equiv \frac{\Pi_i k_i^3}{\sum_i k_i^3} \frac{B_\zeta}{4\pi^4 m_P^2 \mathcal{P}_\zeta^2}. \quad (70)$$

In the context of multi-field inflation, the so-called δN -formalism [9] is particularly useful to evaluate the primordial non-Gaussianity generated on large scales [10]. The idea is to describe, on scales larger than the Hubble radius, the non-linear evolution of perturbations generated during inflation in terms of the perturbed expansion from an initial hypersurface (usually taken at Hubble crossing during inflation) up to a final uniform-density hypersurface (usually during the radiation-dominated era). Using the expansion

$$\zeta \simeq \sum_I N_{,I} \delta\varphi_*^I + \frac{1}{2} \sum_{IJ} N_{,IJ} \delta\varphi_*^I \delta\varphi_*^J \quad (71)$$

yields the expression

$$\begin{aligned} \langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle &= \sum_{IJK} N_{,I} N_{,J} N_{,K} \langle \delta\varphi_{\mathbf{k}_1}^I \delta\varphi_{\mathbf{k}_2}^J \delta\varphi_{\mathbf{k}_3}^K \rangle + \\ &\quad \frac{1}{2} \sum_{IJKL} N_{,I} N_{,J} N_{,K} N_{,L} \langle \delta\varphi_{\mathbf{k}_1}^I \delta\varphi_{\mathbf{k}_2}^J (\delta\varphi_{\mathbf{k}_3}^K \star \delta\varphi_{\mathbf{k}_3}^L) \rangle + \text{perms.} \end{aligned} \quad (72)$$

If the scalar field fluctuations are quasi-Gaussian, one can ignore their three-point correlations and, after substituting

$$\langle \delta\varphi_{\mathbf{k}_1}^I \delta\varphi_{\mathbf{k}_2}^J \rangle = (2\pi)^3 \delta_{IJ} \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \frac{2\pi^2}{k_1^3} \mathcal{P}_*(k_1), \quad \mathcal{P}_*(k) \equiv \frac{H_*^2}{4\pi^2}, \quad (73)$$

one gets

$$\frac{6}{5}f_{\text{NL}} = \frac{N_I N_J N^{IJ}}{(N_K N^K)^2}. \quad (74)$$

The present observational constraints [2] are $-9 < f_{\text{NL}}^{(\text{local})} < 111$ (95% CL) and $-151 < f_{\text{NL}}^{(\text{equil})} < 253$ (95% CL), for respectively, the local non-linear coupling parameter and the equilateral non-linear coupling parameter (characterizing the amplitude of the bispectrum of the equilateral configurations in which the three wave vectors forming a triangle in Fourier space have the same length).

4.3 The curvaton scenario

The curvaton (see [11]) is a weakly coupled scalar field, χ , which is light relative to the Hubble rate during inflation, and hence acquires an almost scale-invariant spectrum and effectively Gaussian distribution of perturbations, $\delta\chi$, during inflation,

$$\mathcal{P}_{\delta\chi} = \left(\frac{H}{2\pi}\right)^2. \quad (75)$$

After inflation the Hubble rate drops and eventually the curvaton becomes non-relativistic so that its energy density grows relative to radiation, until it contributes a significant fraction of the total energy density, $\Omega_\chi \equiv \bar{\rho}_\chi/\bar{\rho}$, before it decays. Hence the initial curvaton field perturbations on large scales can give rise to a primordial density perturbation after it decays.

The non-relativistic curvaton (mass $m \gg H$), before it decays, can be described by a pressureless, non-interacting fluid with energy density

$$\rho_\chi = m^2 \chi^2, \quad (76)$$

where χ is the rms amplitude of the curvaton field, which oscillates on a timescale m^{-1} much less than the Hubble time H^{-1} . The corresponding perturbations are characterized, using (41) and (75),

$$\zeta_\chi = \left(\frac{\delta\rho_\chi}{3\rho_\chi}\right)_{\text{flat}} \Rightarrow \mathcal{P}_{\zeta_\chi} \simeq \frac{H^2}{9\pi^2\chi^2}. \quad (77)$$

When the curvaton decays into radiation, its perturbations are converted into perturbations of the resulting radiation fluid. The subsequent perturbation is described by

$$\zeta_r = r\zeta_\chi + (1-r)\zeta_{\text{inf}}, \quad r \equiv \frac{3\Omega_{\chi,\text{decay}}}{4 - \Omega_{\chi,\text{decay}}}. \quad (78)$$

This implies that the power spectrum for the primordial adiabatic perturbation ζ_r can be expressed as

$$\mathcal{P}_{\zeta_r} = \mathcal{P}_{\zeta_{\text{inf}}} + r^2\mathcal{P}_{\zeta_\chi}. \quad (79)$$

where $\mathcal{P}_{\zeta_{\text{inf}}}$ is given by (58) in the case of standard single field inflation. In most cases, the inflaton contribution is supposed to be negligible but one can

also envisage mixed inflaton-curvaton scenarios where both contribute (see e.g. [12]).

Interestingly, the curvaton scenario can give rise to a significant non-Gaussianity of the local type, since the expression (74) yields [10]

$$f_{NL}^{\text{local}} = \frac{5}{4r} - \frac{5}{3} - \frac{5}{6}r. \quad (80)$$

The curvaton can also produce some isocurvature perturbations [13], possibly with a significant non-Gaussianity [14].

4.4 Multi-inflaton scenario

We now consider multi-field models, which can be described by an action of the form

$$S = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} + P(X^{IJ}, \phi^K) \right] \quad (81)$$

where P is an arbitrary function of N scalar fields and of the kinetic term

$$X^{IJ} = -\frac{1}{2} \nabla_\mu \phi^I \nabla^\mu \phi^J. \quad (82)$$

The very general form (81) can be seen as an extension of k-inflation [15] to the case of several scalar fields.

A more restrictive class of models, considered in [16], consists of Lagrangians that depend on a global kinetic term $X = G_{IJ} X^{IJ}$ where $G_{IJ} \equiv G_{IJ}(\phi^K)$ is an arbitrary metric on the N -dimensional field space. By defining $P = X - V$, one recovers in particular multi-field models with an action of the form

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} G_{IJ}(\phi) \partial^\mu \phi^I \partial_\mu \phi^J - V(\phi) \right), \quad (83)$$

where a flat metric in field space ($G_{IJ} = \delta_{IJ}$) corresponds to standard kinetic terms.

The relations obtained in the previous section for the single field model can then be generalized. The energy-momentum tensor, derived from (81), is of the form

$$T^{\mu\nu} = P g^{\mu\nu} + P_{<IJ>} \partial^\mu \phi^I \partial^\nu \phi^J, \quad (84)$$

where $P_{<IJ>}$ denotes the partial derivative of P with respect to X^{IJ} (symmetrized with respect to the indices I and J). The equations of motion for the scalar fields, which can be seen as generalized Klein-Gordon equations, are obtained from the variation of the action with respect to ϕ^I . One finds

$$\nabla_\mu (P_{<IJ>} \nabla^\mu \phi^J) + P_{,I} = 0. \quad (85)$$

where $P_{,I}$ denotes the partial derivative of P with respect to ϕ^I .

In a spatially flat FLRW spacetime, with metric

$$ds^2 = -dt^2 + a^2(t) d\vec{x}^2, \quad (86)$$

the scalar fields are homogeneous, so that $X^{IJ} = \dot{\phi}^I \dot{\phi}^J / 2$, and the energy-momentum tensor reduces to that of a perfect fluid with energy density

$$\rho = 2P_{<IJ>} X^{IJ} - P, \quad (87)$$

and pressure P . The evolution of the scale factor $a(t)$ is governed by the Friedmann equations, which can be written in the form

$$H^2 = \frac{1}{3} (2P_{<IJ>} X^{IJ} - P), \quad \dot{H} = -X^{IJ} P_{<IJ>}. \quad (88)$$

The equations of motion for the scalar fields reduce to

$$\left(P_{<IJ>} + P_{<IL>,<JK>} \dot{\phi}^L \dot{\phi}^K \right) \ddot{\phi}^J + \left(3H P_{<IJ>} + P_{<IJ>,<K>} \dot{\phi}^K \right) \dot{\phi}^J - P_{,I} = 0, \quad (89)$$

where $P_{<IL>,<JK>}$ denotes the (symmetrized) second derivative of P with respect to X^{IL} and X^{JK} .

The expansion up to second order in the linear perturbations of the action (81) is useful to obtain the classical equations of motion for the perturbations and to calculate the spectra of the primordial perturbations generated during inflation, as we have seen in the previous section for a single scalar field. Working for convenience with the scalar field perturbations Q^I defined in the spatially flat gauge, the second order action can be written in the rather simple form

$$S_{(2)} = \frac{1}{2} \int dt d^3x a^3 \left[(P_{<IJ>} + 2P_{<MJ>,<IK>} X^{MK}) \dot{Q}^I \dot{Q}^J - P_{<IJ>} h^{ij} \partial_i Q^I \partial_j Q^J - \mathcal{M}_{KL} Q^K Q^L + 2\Omega_{KI} Q^K \dot{Q}^I \right] \quad (90)$$

where the explicit expressions for the mass matrix \mathcal{M}_{KL} and for the mixing matrix Ω_{KI} can be found in [17].

4.5 Example: multi-field DBI inflation

Recent years have seen an intensive effort to construct models of inflation within string theory (for a recent review, see e.g. [18]). Effective descriptions of string theory at low energies are based on 10-dimensional spacetimes, which can be related to our apparent 4-dimensional spacetime by assuming that six dimensions span a 6-dimensional compact internal manifold. In addition to the fundamental strings, string theories contain higher dimensional objects called D-branes (where the D stands for Dirichlet boundary conditions). An interesting suggestion was to identify the inflaton(s) with the position of some D-brane in the internal compact space. Effective four-dimensional inflation could thus result from the motion of a D-brane in the internal dimensions. This type of scenario is called *brane inflation* [19].

Let us consider, for instance, a D3-brane with tension T_3 evolving in a 10-dimensional geometry described by the metric

$$ds^2 = h^{-1/2}(y^K) g_{\mu\nu} dx^\mu dx^\nu + h^{1/2}(y^K) G_{IJ}(y^K) dy^I dy^J \equiv H_{AB} dY^A dY^B \quad (91)$$

with coordinates $Y^A = \{x^\mu, y^I\}$, where $\mu = 0, \dots, 3$ and $I = 1, \dots, 6$.

The motion of the brane is described by a Dirac-Born-Infeld Lagrangian,

$$L = -T_3 \sqrt{-\det \gamma_{\mu\nu}} \quad (92)$$

which depends on the determinant of the induced metric on the 3-brane,

$$\gamma_{\mu\nu} = H_{AB} \partial_\mu Y_{(b)}^A \partial_\nu Y_{(b)}^B = h^{-1/2} (g_{\mu\nu} + h G_{IJ} \partial_\mu \varphi^I \partial_\nu \varphi^J), \quad (93)$$

where the functions $Y_{(b)}^A(x^\mu) = (x^\mu, \varphi^I(x^\mu))$ define the brane embedding (with the x^μ being the spacetime coordinates on the brane). After various rescalings, one ends up with a Lagrangian of the form

$$P = -\frac{1}{f(\phi^I)} (\sqrt{\mathcal{D}} - 1) - V(\phi^I) \quad (94)$$

with

$$\mathcal{D} \equiv \det(\delta_\nu^\mu + f G_{IJ} \partial^\mu \phi^I \partial_\nu \phi^J), \quad (95)$$

and where we have also included potential terms, which arise from the brane's interactions with bulk fields or other branes.

An interesting situation is when the brane moves in a higher dimensional warped conical geometry, along the radial direction. If one ignores the angular internal coordinates, the four-dimensional effective action reduces to

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{f} (\sqrt{1 + f \partial_\mu \phi \partial^\mu \phi} - 1) - V(\phi) \right], \quad (96)$$

which depends on a single scalar field but with non standard kinetic terms. This action belongs to the class of k -inflation models [15] characterized by a Lagrangian of the form $P(X, \phi)$, where $X = -\partial_\mu \phi \partial^\mu \phi / 2$. If $f \dot{\phi}^2 \ll 1$, one can expand the square root in the Lagrangian and one recovers the usual kinetic term familiar to slow-roll inflation. But there is another regime, called DBI inflation [20], corresponding to the “relativistic” limit

$$1 - f \dot{\phi}^2 \ll 1 \Rightarrow |\dot{\phi}| \simeq 1/\sqrt{f}, \quad (97)$$

which does not require a very flat potential as in standard slow-roll inflation.

Allowing the brane to move in the angular directions leads to a multi-field scenario, since each brane coordinate in the extra dimensions gives rise to a scalar field from the effective four-dimensional point of view. The corresponding Lagrangian (94) can be written in the generic form (81) that depends on the kinetic terms X^{IJ} defined in (82) by noting that [21]

$$\begin{aligned} \mathcal{D} &= 1 - 2f G_{IJ} X^{IJ} + 4f^2 X_I^{[I} X_J^{J]} - 8f^3 X_I^{[I} X_J^J X_K^{K]} + 16f^4 X_I^{[I} X_J^J X_K^K X_L^{L]} \\ &\equiv 1 - 2f \tilde{X} \end{aligned} \quad (98)$$

where the field indices are lowered by the field metric G_{IJ} .

The dynamics of the linear perturbations can be obtained from the general expressions (90). Alternatively, one can use the results of [16] for Lagrangians of the form $P = P(X, \phi^K)$, where $X = G_{IJ}X^{IJ}$, by writing the Lagrangian (94) as a function of \tilde{X} , introduced in (98), so that $P(X^{IJ}, \phi^K) = \tilde{P}(\tilde{X}, \phi^K)$ (note that \tilde{X} and X coincide in the homogeneous background). What characterizes the DBI multi-field Lagrangian is that all linear perturbations propagate with a common velocity, namely the effective speed of sound defined by

$$c_s = \sqrt{1 - 2f\tilde{X}}. \quad (99)$$

For simplicity, let us now concentrate on a two-field scenario. It is then useful to decompose the scalar field perturbations into adiabatic and entropic modes [22], namely

$$Q^I = Q_\sigma e_\sigma^I + Q_s e_s^I, \quad (100)$$

where

$$e_\sigma^I = \frac{\dot{\phi}^I}{\sqrt{2\tilde{X}}}, \quad (101)$$

is the unit vector along the inflationary trajectory in field space and the entropy vector e_s^I is the unit vector orthogonal to the adiabatic vector e_σ^I , i.e.

$$G_{IJ}e_s^I e_s^J = 1, \quad G_{IJ}e_s^I e_\sigma^J = 0. \quad (102)$$

As in standard inflation discussed in the previous section, it is more convenient, after going to conformal time $\tau = \int dt/a(t)$, to work in terms of the canonically normalized fields

$$v_\sigma = \frac{a}{c_s^{3/2}} Q_\sigma, \quad v_s = \frac{a}{\sqrt{c_s}} Q_s, \quad (103)$$

which lead to the second order action

$$\begin{aligned} S_{(2)} = & \frac{1}{2} \int d\tau d^3x \left\{ v_\sigma'^2 + v_s'^2 - 2\xi v_\sigma' v_s - c_s^2 [(\partial v_\sigma)^2 + (\partial v_s)^2] \right. \\ & \left. + \frac{z''}{z} v_\sigma^2 + \left(\frac{\alpha''}{\alpha} - a^2 \mu_s^2 \right) v_s^2 + 2 \frac{z'}{z} \xi v_\sigma v_s \right\} \end{aligned} \quad (104)$$

with

$$\xi = \frac{a}{\dot{\tilde{P}}_{,X} c_s} [(1 + c_s^2) \tilde{P}_{,s} - c_s^2 \dot{\sigma}^2 \tilde{P}_{,Xs}], \quad \dot{\sigma} \equiv \sqrt{2\tilde{X}}, \quad (105)$$

and where we have introduced the two background-dependent functions $z = a\dot{\sigma}/(Hc_s^{3/2})$ and $\alpha = a/\sqrt{c_s}$.

The equations of motion derived from the action (104) can be written in the compact form

$$v_\sigma'' - \xi v_s' + \left(k^2 c_s^2 - \frac{z''}{z} \right) v_\sigma - \frac{(z\xi)'}{z} v_s = 0. \quad (106)$$

$$v_s'' + \xi v_\sigma' + \left(k^2 c_s^2 - \frac{\alpha''}{\alpha} + a^2 \mu_s^2 \right) v_s - \frac{z'}{z} \xi v_\sigma = 0. \quad (107)$$

Assuming that the coupling ξ is very small and that $|\mu_s^2|/H^2 \ll 1$ when the scales of interest cross out the sound horizon, i.e. $kc_s = aH$, the above system leads to the amplification of the vacuum fluctuations at sound horizon crossing for both adiabatic and entropic degrees of freedom.

Following the standard procedure outlined in the previous section, one selects the positive frequency solutions of Eqs. (106) and (107), which correspond to the usual vacuum on very small scales:

$$v_{\sigma k} \simeq v_{s k} \simeq \frac{1}{\sqrt{2kc_s}} e^{-ikc_s\tau} \left(1 - \frac{i}{kc_s\tau}\right). \quad (108)$$

As a consequence, the power spectra for v_σ and v_s after sound horizon crossing have the same amplitude

$$\mathcal{P}_{v_\sigma} = \mathcal{P}_{v_s} = \frac{k^3}{2\pi^2} |v_{\sigma k}|^2 \simeq \frac{H^2 a^2}{4\pi^2 c_s^3}. \quad (109)$$

However, in terms of the initial fields Q_σ and Q_s , one finds, using (103),

$$\mathcal{P}_{Q_{\sigma*}} \simeq \frac{H^2}{4\pi^2}, \quad \mathcal{P}_{Q_{s*}} \simeq \frac{H^2}{4\pi^2 c_s^2}, \quad (110)$$

(the subscript $*$ indicates that the corresponding quantity is evaluated at sound horizon crossing $kc_s = aH$) which shows that, for small c_s , the entropic modes are *amplified* with respect to the adiabatic modes:

$$Q_{s*} \simeq \frac{Q_{\sigma*}}{c_s}. \quad (111)$$

In order to confront the predictions of inflationary models to cosmological observations, it is useful to rewrite the scalar field perturbations in terms of geometrical quantities. Using the relation

$$\mathcal{R} = \frac{H}{\dot{\sigma}} Q_\sigma, \quad (112)$$

one recovers the usual *single-field* result [23] that the power spectrum for \mathcal{R} at sound horizon crossing is given by

$$\mathcal{P}_{\mathcal{R}*} = \frac{k^3}{2\pi^2} \frac{|v_{\sigma k}|^2}{z^2} \simeq \frac{H^4}{4\pi^2 \dot{\sigma}^2} = \frac{H^2}{8\pi^2 \epsilon c_s}, \quad (113)$$

where $\epsilon = -\dot{H}/H^2$.

However, in contrast with single field inflation, the curvature perturbation can be subsequently modified if there is a transfer between entropy and adiabatic modes [24] (see also [25] for a recent numerical treatment). This transfer from the entropic to the adiabatic modes can be parametrized by the transfer coefficient which appears in the formal solution $\mathcal{R} = \mathcal{R}_* + T_{\mathcal{R}\mathcal{S}} \mathcal{S}_*$ of the evolution equations. For convenience, we use the entropy perturbation, which we

denote \mathcal{S} , whose power spectrum at sound horizon crossing is the same as that of the curvature perturbation, i.e.

$$\mathcal{S} = c_s \frac{H}{\dot{\sigma}} Q_s, \quad (114)$$

so that $\mathcal{P}_{\mathcal{S}} = \mathcal{P}_{\mathcal{R}_*}$. The final curvature power-spectrum is thus given by

$$\mathcal{P}_{\mathcal{R}} = (1 + T_{\mathcal{R}\mathcal{S}}^2) \mathcal{P}_{\mathcal{R}_*} = \frac{\mathcal{P}_{\mathcal{R}_*}}{\cos^2 \Theta}, \quad (115)$$

where we have introduced the “transfer angle” Θ ($\Theta = 0$ if there is no transfer and $|\Theta| = \pi/2$ if the final curvature perturbation is mostly of entropic origin) by

$$\sin \Theta = \frac{T_{\mathcal{R}\mathcal{S}}}{\sqrt{1 + T_{\mathcal{R}\mathcal{S}}^2}}. \quad (116)$$

The power spectrum for the tensor modes is still governed by the transition at *Hubble radius* and its amplitude, given by (63), is much smaller than the curvature amplitude in the small c_s limit. The tensor to scalar ratio is

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_{\mathcal{R}}} = 16 \epsilon c_s \cos^2 \Theta. \quad (117)$$

Interestingly this expression combines the result of k -inflation, where the ratio is suppressed by the sound speed c_s , and that of standard multi-field inflation[26].

It is also possible to compute the non-Gaussianities generated in these models. For multi-field DBI inflation, the shape of non-Gaussianities is found to be the same as in single-field DBI but their amplitude is affected by the transfer between the entropic and adiabatic modes. The contribution from the scalar field three-point functions to the coefficient f_{NL} is given by [21]

$$f_{NL}^{(3)} = -\frac{35}{108} \frac{1}{c_s^2} \frac{1}{1 + T_{\mathcal{R}\mathcal{S}}^2} = -\frac{35}{108} \frac{1}{c_s^2} \cos^2 \Theta. \quad (118)$$

The effect of entropy modes is therefore potentially important in the perspective of confronting DBI models to future CMB observations.

To conclude, multi-field inflation is a very rich playground, where entropy modes can play a significant role. The most important consequence of entropy modes is the possibility to modify the curvature perturbation, on large scales, in contrast with single field inflation. This means that the adiabatic fluctuations, which we observe today in the CMB, could come originally from entropy perturbations produced during multi-field inflation.

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